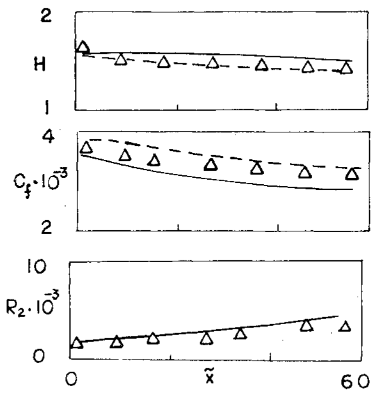

Fig. 4 Integral parameters of the boundary layer; $|\alpha| = 3$, $|\beta| = 3$, $|\gamma| = 0.01$; ——, present predictions; --, Ref. 6; Δ, Ref. 9.

shape factor H, friction coefficient C_f, and the momentum thickness Reynolds number are shown as functions of the streamwise coordinate $\tilde{x}$. The shape factor remains almost constant, whereas the friction coefficient is slightly underpredicted.

Summary

The main features of this work are summarized as follows.

1) With the inclusion of second-order (gradient diffusion) terms for $\tilde{k}$, $\widetilde{uv}$, and $\tilde{L}$, the method is capable of predicting closely the mean flow as well as the turbulent characteristics of the flat-plate boundary layer.

2) In the case when $|\alpha|$, $|\beta|$, and $|\gamma|$ are greater than zero, stability is restored because of the condition that $\tilde{k}$, $\widetilde{uv}$, and $\tilde{L}$ appear under second-order derivatives, and the system of equations becomes parabolic. This supports the earlier idea that for the hyperbolic set of equations the well-posed character of the differential equations is altered in a region close to the inner boundary.[1]

3) The restoration of numerical stability for $|\alpha|$, $|\beta|$, and $|\gamma|$ greater than zero and the values that they assume suggest that gradient diffusion should be accounted for at least close to the inner boundary.

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Generalized Area Rules and Integral Theorems for Hypersonic Wing-Bodies

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Introduction

THE advent of modern shock-capturing techniques has led to nonlinear flow solutions for complex configurations such as the Shuttle Orbiter. A description of a typical implementation is discussed in Ref. 1. Recent interest in hypersonic transport and scramjet vehicles employing favorable interference as well as high-acceleration missiles has stimulated the need for increased understanding of the parametric dependencies in such flows. Generally, these relationships are not readily accessible from computer solutions. In particular, the compression lift on flat-top wing-body combinations provides a basis for obtaining high aerodynamic efficiencies. Analytical results are required to clarify the geometrical relationships necessary to optimize these benefits.

In Ref. 2, an area rule was obtained for the change in L/D of a hypersonic delta wing due to the addition on its compressive side of a conical body of arbitrary cross section. The body was considered *conically subsonic*, and the details of the pressure field over that configuration were quantified in Ref. 3. In this Note, the appropriate relationships will be obtained for a *nonconical* body and a *conically supersonic* conical shape. Finally, area progressions will be derived optimizing L/D for the nonconical class of configurations.

Nonconical Flows

Figure 1 depicts a nonconical body $OABA'$ on the compressive side of a hypersonic delta wing OCC' in the plane $\bar{y}=0$. The equation of the body is $\bar{y}=\epsilon\delta f(\bar{x},\bar{z})$, $-z_B \leq z \leq z_B(\bar{x})$, where $z=\pm z_B(x)$ are the "ridge line" curves OA and OA', and f is a continuous single-valued function with $f(\bar{x}, \pm z_B)=0$.

The wing-body is assumed to be weakly three dimensional in the sense indicated in Ref. 2, through a limit involving the sweepback angle χ. The nonconical problem for the pressure perturbation about the freestream field is hyperbolic, and the domain of dependence of the body depends on whether $z_B(\bar{x})$ is time-like or space-like[4] or correspondingly subsonic or supersonic in the sense that the component of the local Mach number normal to the edge is subsonic or supersonic. Associated with the former case, $z_B'(\bar{x}) < \tan\mu$, where μ, the local Mach angle, is given by $\sin^{-1}c\delta$, and $c^2=\gamma(\gamma-1)/2$, where γ is the specific heat ratio.† For supersonic edges, the inequality is reversed. The nonconical results derived herein are restricted to the subsonic leading-edge case shown in Fig. 1, where the body is assumed to have a conical nose and the domain of dependence is the Mach cone $z^2=c^2x^2-y^2$ $(x=\bar{x}$, $y=\bar{y}/\delta$, $z=\bar{z}/\delta)$. It is plausible that the results to be derived hold for the supersonic leading-edge case in which the domain of dependence is bounded by a characteristic envelope of Mach cones whose apices lie along OA and OA'. This conclusion relates to superposition procedures that can be used

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†The freestream Mach number, M_∞, is assumed hereinafter infinite for convenience. Generalization to arbitrary hypersonic M_∞'s is straightforward.

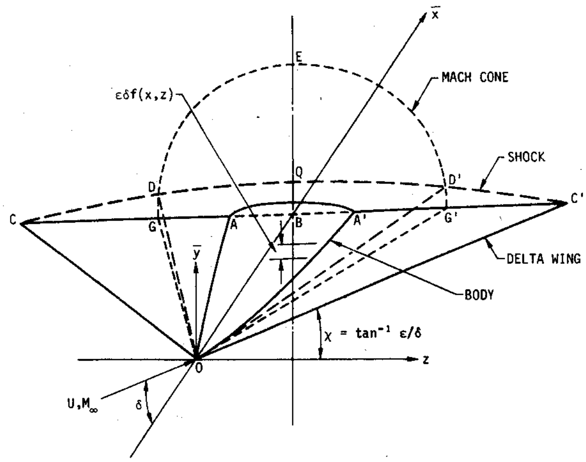

Fig. 1 Delta wing-body configuration.

after a local application of the present results to each element of the envelope.

Corresponding to the delta wing of Fig. 1, the region between the Mach cone $z_I(x,y)$ and the shock, $\{0 \leq y \leq [(\gamma-1)/2]x\}$ is constant state.

The solution for the flow over the wing-body can be obtained as a superposition of two problems. In the first, the wing sweepback is zero, and in the second, the body thickness f vanishes. In all other respects, both problems are identical. Therefore, the first problem gives the body contribution, p, to the wing-body pressure field. It can be shown that this quantity is governed by the wave equation

$$c^{-2}p_{xx} - p_{yy} - p_{zz} = 0 \tag{1a}$$

In an infinite Mach number specialization of the limit described in Ref. 2, the boundary conditions can be satisfied on the basic planar shock $\bar{S}$: $y = (\gamma-1)x/2$ and the wing $y=0$. On the shock,

$$D_S p \equiv 3p_{xx} + (5\gamma - 1)p_{xy} + \gamma(2\gamma - 1)p_{yy} = 0 \tag{1b}$$

On the body,

$$p_y(x,0,z) = -\frac{2}{\gamma-1} f_{xx} \tag{1c}$$

Defining the spanwise pressure integral Q as

$$Q = \int_0^{z_I} p \, \mathrm{d}z$$

integration of Eqs. (1) and by parts manipulations gives

$$(\partial/\partial x^2 - c^2 \partial^2/\partial y^2)Q = 0 \tag{2a}$$

$$D_S Q = 0 \text{ on } \bar{S} \tag{2b}$$

$$Q_y(x,0) = -[2/(\gamma-1)]S''(x) \tag{2c}$$

where the semicross-sectional area S is defined by

$$S(x) = \int_0^{z_B} f \mathrm{d}z$$

and the fact that the surface $z_I(x,y)$ is characteristic carrying constant pressure has been used to eliminate boundary terms in Eqs. (2a) and (2b) arising from differentiation of the upper limit of integration, z_I. Implicit in the argument is analyticity of p at the triple point intersection of the Mach cone and shock (points D and D' in Fig. 1) which can be demonstrated on the basis of the boundary conditions. A more detailed discussion of these steps is presented in Ref. 5. Hence, Q can be determined by methods similar to those used in Ref. 6. Accordingly, Eq. (2a) has the general solution

$$Q = F(x+y/c) + G(x-y/c)$$

From Eq. (2c), this can be written as

$$Q = F(x+y/c) + F(x-y/c) + Y_S^{-1}S'(x-y/c)$$

where $Y_S \equiv c/\gamma$, and the prime denotes differentiation with respect to the arguments. Substitution into Eq. (2b) gives the following functional equation for F:

$$F''(t) + \beta F''(kt) = -Y_S^{-1}\beta S'''(kt)$$

$$t \equiv (1+Y_S)x, \quad k \equiv (1-Y_S)/(1+Y_S)$$

$$\beta k \equiv (2\gamma - 1 - 3c)/(2\gamma - 1 + 3c) \tag{3}$$

The solution of Eq. (3) is:

$$F''(t) = Y_S^{-1} \sum_{n=1}^{\infty} (-\beta k^2)^n S'''(k^n t)$$

or

$$F(t) = -Y_S^{-1} \sum_{n=1}^{\infty} (\beta/k^2)^n S'(k^2 t) + C_1 t + C_2$$

On substitution of this expression into the previous relation for Q, we obtain

$$Y_S Q = S'(x-y/c) + \sum_{n=1}^{\infty} (-\beta)^n [S'(k^n(x+y/c)) + S'(k^n(x-y/c))] + a_1 x + a_2 \tag{4}$$

Corresponding to the assumption of conical noses, ($S'(0)=0$), and the definition of Q, $Q(0,0)=0$. Evaluation of Eq. (4) at $x=0$ with these relations implies that $a_2=0$. Since $S(0)=0$, we have $S(x) = x^2 S''(0)/2$ as $x \to 0$, and by virtue of the relations proven in Ref. 2, $Q_x(0,0) = 3\gamma S''(0)/(2\gamma-1)$, which with Eq. (4) as detailed in Ref. 5 leads to $a_1 = 0$ for $S''(0) \neq 0$.

Defining a normal force integral N as

$$N = \int_0^1 Q(x,0)\mathrm{d}x$$

Eq. (4) implies

$$Y_S N = S(1) + 2\sum_{n=1}^{\infty} (-\beta/k)^n S(k^n) \tag{5}$$

Eq. (5) is the hypersonic area rule for nonconical bodies and states that the increment in lift of a delta wing due to body addition depends on an area progression at successive reflections of a two-dimensional disturbance emanating from the body, $y=0$, with the shock $y = [(\gamma-1)/2]x$.

Since $S = O(1)$, there should exist a μ for locally conical noses as $x \to 0$, such that

$$S(x) \leq \mu x^2 \text{ for } 0 \leq x \leq 1$$

An upper bound for the sum in Eq. (5) is, therefore,

$$\mu \sum_{n=1}^{\infty} |\beta k|^n = \mu|\beta k|/(1 - |\beta k|)$$

which gives a relative maximum N

$$Y_S N = 1 + 2\mu\,|\beta k|/(1-|\beta k|)$$

For $1<\gamma<2$, this is achieved if

$$S(k^n) = (-1)^n \mu k^{2n}$$

This expression represents a three-dimensional generalization of the result presented in Ref. 7, in which beneficial interference is achieved by a corrugated shape to achieve optimal pressures on favorably inclined surfaces.

"Supercritical" Conical Flows

In contrast to the previous section, we consider herein the case of supersonic ridge lines in a restricted context of conical flows. We, therefore, treat the special class of shapes for which $f=xF(Z)$, $p=p(Y,Z)$, ($Y\equiv y/cx$ and $Z\equiv z/cx$). Defining Z_R as the coordinate of the ridge line OA' in Fig. 1, then if $Z_R>1$, OA' is supersonic and the conical projection of the flow can be represented as shown in Fig. 2, where the unit circle is the Mach cone inside which the flow is conically subsonic. Outside this boundary, the field is conically supersonic. In analogy to Ref. 2,‡ a generalized spanwise integral P is defined as

$$P(Y) = \int_0^{Z_l} {}^{+} p\,\mathrm{d}z$$

where $Z_l \equiv Z_R - \sqrt{Z_R^2-1}\;Y$, ($Z_R=$const), with the plus sign superscript signifying integration to a point Z slightly greater than Z_l as before. The plane $Z=Z_l(Y)$ denotes the position of a leading Mach wave forming an upstream boundary of the hyperbolic zone. Using methods similar to those of Ref. 2, which are detailed in Ref. 5, and the fact that the characteristic Z_l borders a constant-state region, the variation of P with Y can be derived as

$$P = \frac{1}{2\gamma-1}\left[3\gamma V-(\gamma-1)c\right] + p_l\left(Z_T - \frac{c}{\gamma}B\right) + \left(p_l B - \frac{\gamma}{c}V\right)Y \tag{6}$$

where p_l is the pressure immediately to the right of the leading characteristic, $B\equiv -\sqrt{Z_R^2-1}$, $Z_T\equiv Z_R-\sqrt{Z_R^2-1}\;Y_S$, and

$$V \equiv 2\int_0^{Z_R} F(Z)\,\mathrm{d}Z \tag{7}$$

The $Y=0$ specialization of Eq. (6) constitutes the conically supersonic extension of the subsonic leading-edge area rule derived in Ref. 2 where the incremental loading due to body addition on a delta wing is obtained from Eq. (6).

Discussion

Equation (6) specializes to the result given in Ref. 2 for the corresponding vanishing value of p_l. This condition is obtained not only for conically subsonic bodies, but for supersonic ones as well, since the evaluation of p is to be taken ahead of the secondary wave, which is consistent with the plus sign on the upper limit in the definition of P. If

$$1 \le Z_R \le \sqrt{(\gamma+1)/2\gamma}$$

then the hyperbolic region is shown as DRR' in Fig. 3, and is a simple wave in the sense that the pressure is constant along

‡In contrast to the preceding sections, here the quantity p signifies the pressure field over a complete delta-wing body of the type treated in Ref. 2, and is therefore a superposition of the thickness and wing-alone parts described prior to Eq. (1a).

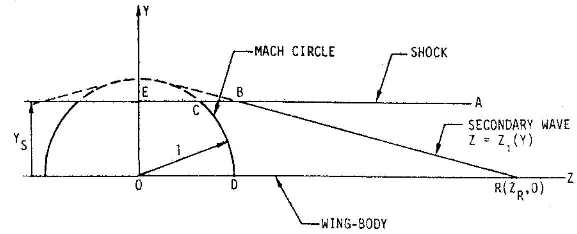

Fig. 2 Domains of conically supersonic wing-body interaction problem.

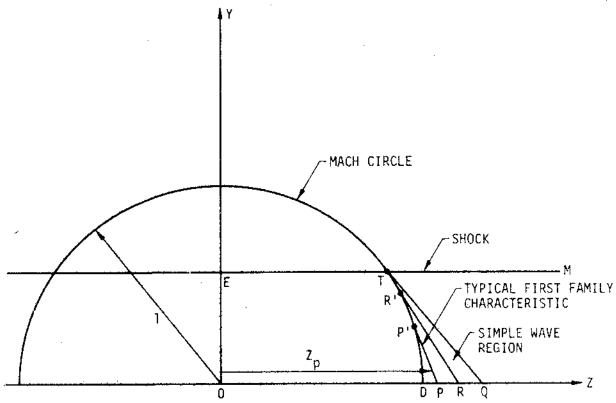

Fig. 3 Conical projection of simple wave configuration.

first-family characteristics such as PP'. The conically supersonic pressure field is thereby uniquely determined from the boundary data on DR. For

$$Z_R \ge \sqrt{(\gamma+1)/2\gamma}$$

reflections of the opposite family from the shock ETM interact with this primary flow. The associated pressure field will be described in Ref. 8. If the "partial span load" is defined as

$$P_p = \int_0^{Z_p} p\,\mathrm{d}Z$$

and Z_p measures the distance to an intermediate characteristic for $0\le Y\le Y_p$ in Fig. 3 or the Mach cone for $Y_p\le Y\le Y_S$, a generalization of the foregoing analysis may be possible to obtain P_p. The crux of the argument depends on whether the simple wave variation of pressure on the segment of the Mach circle $P'R'$ can be used to eliminate or calculate boundary terms for generalized conical counterparts of Eqs. (2a) and (1a). In this connection, we recall that the results obtained in this paper are restricted to $p_l=$const. Evaluation of P_p could serve as a useful validation of the complete pressure field.

It is interesting to note that Eq. (6) is similar in certain respects to Hayes' equivalence principle[9] for linearized supersonic flows. For the latter, the far-field points see only line source equivalents of finite area distributions in the near field. This is satisfied in the present case since Z_R is bounded. These results have a cross connection with the validity of the area rule to nonconical flows. In fact, it can be shown by setting $S(x)/S(1)=x^2$ that Eq. (5) reduces to Eq. (6).

Acknowledgment

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Technical Comments

Comment on "Estimation of Fundamental Frequencies of Beams and Plates with Varying Thickness"

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EASTEP[1] has presented a perturbation theory for determining the frequencies and normal modes of beams and plates with variable thickness. All of Eastep's results can be obtained more directly from the well-known classical perturbation theory, first set forth in systematic form by Rayleigh[2] and since then treated with varying degrees of mathematical rigor in many texts on physics, mathematics, and vibration theory.[3-6] Rayleigh's original work is not only general in form but most often leads to the desired results in the most straightforward fashion, as will be illustrated.

The procedure used by Eastep, in which the perturbation process is developed ab initio for each problem utilizing the specific differential (or other) equations of the problem, is also well known. (Such an approach is needed when all of the frequencies and normal modes of the unperturbed problem are not known. See Ref. 7 for an illustrative example of this approach.) In fact, a development exactly equivalent to Eastep's Eqs. (5-10) for a tapered beam was given in the text by Morse (Ref. 5, pp. 164-166) except that the boundary conditions of the beam were left general; to get Eastep's equations, it is only necessary to substitute sine functions as the unperturbed normal modes of a simply supported beam of uniform cross-section in the integrals given by Morse.

The remarkable accuracy shown by Eastep for second-order perturbation theory in the case of the linearly tapered beam and the analogous plate problem is interesting, especially in view of the large thickness perturbations involved (thickness variations up to 90% of the maximum thickness). However, such accuracy is not at all typical of second-order perturbation theory in general and is a result of the special characteristics of the problems considered by Eastep, some of which will be discussed below.

Rayleigh's formulation of perturbation theory[2] begins from the equations of motion of the unperturbed system in terms of generalized coordinates q_n corresponding to the normal modes of vibration. These modes are, by definition, uncoupled from one another; in general, the perturbations couple the uncoupled normal modes. The kinetic energy, $T_0+\delta T$, and, the potential energy, $V_0+\delta V$, of the perturbed system, where the δ refer to the effects of the perturbation, are given in terms of the q_n as

$$T_0+\delta T=\tfrac{1}{2}(a_{1_0}+\delta a_{11})\dot{q}_1^2+\tfrac{1}{2}(a_{2_0}+\delta a_{22})\dot{q}_2^2 + \ldots + \delta a_{12}\dot{q}_1\dot{q}_2+\delta a_{13}\dot{q}_1\dot{q}_3+\ldots$$

$$V_0+\delta V=\tfrac{1}{2}(c_{1_0}+\delta c_{11})q_1^2+\tfrac{1}{2}(c_{2_0}+\delta c_{22})\dot{q}_2^2 + \ldots\delta c_{12}q_1q_2+\delta c_{13}q_1q_3+\ldots \tag{1}$$

In relation to the generalized coordinates, it is apparent here that the a are the generalized masses and the c are the generalized stiffnesses. The equations of motion follow from the substitution of $T_0+\delta T$ and $V_0+\delta V$ into Lagrange's equations, resulting in the nth equation of motion at frequency ω,

$$[-(a_{n_0}+\delta a_{nn})\omega^2+(c_{n_0}+\delta c_{nn})]q_n + \sum_{m\neq n}[-\delta a_{nm}\omega^2+\delta c_{nm}]q_m=0 \tag{2}$$

From these equations, Rayleigh's perturbation calculation leads to the perturbation of the nth normal mode to the first order and the nth natural frequency to at least the second order as

$$\frac{q_m}{q_n}=\frac{\omega_{n0}^2\delta a_{nm}-\delta c_{nm}}{a_{m_0}(\omega_{m_0}^2-\omega_{n_0}^2)} \tag{3}$$

and

$$\omega_n^2=\frac{c_{n_0}+\delta c_{nn}}{a_{n_0}+\delta a_{nn}}-\sum_{m\neq n}\frac{(\delta c_{nm}-\omega_{n0}^2\delta a_{nm})^2}{a_{m_0}a_{n_0}(\omega_{m_0}^2-\omega_{n_0}^2)} \tag{4}$$

The development in terms of powers of a perturbation parameter requires expression of the δc_{nm} and δa_{nm} in powers of that parameter and expansion of the denominator of the first terms on the left in a power series as

$$\frac{c_{n0}+\delta c_{nn}}{a_{n_0}+\delta a_{nn}}=\frac{(c_{n0}+\delta c_{nn})}{a_{n_0}}\left[1-\frac{\delta a_{nn}}{a_{n_0}}+\left(\frac{\delta a_{nn}}{a_{n_0}}\right)^2\ldots\right] \tag{5}$$

However, the term prior to expansion can be recognized as the Rayleigh quotient for the perturbed system calculated on the basis of unperturbed normal modes.

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